On the renormalization of the two-point Green function in the sine-Gordon model

2006 J. Phys. A: Math. Gen. 39 2177

(http://iopscience.iop.org/0305-4470/39/9/013)

Institute of Physics Publishing Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **39** (2006) 2177–2201 doi:10.1088/0305-4470/39/9/013

On the renormalization of the two-point Green function in the sine-Gordon model

H Bozkaya, M Faber, A N Ivanov[1] and M Pitschmann

Atominstitut der Österreichischen Universitäten, Arbeitsbereich Kernphysik und Nukleare Astrophysik, Technische Universität Wien, Wiedner Hauptstrasse 8-10/142, A-1040 Wien, Österreich

E-mail: hidir@kph.tuwien.ac.at, faber@kph.tuwien.ac.at, ivanov@kph.tuwien.ac.at and pitschmann@kph.tuwien.ac.at

Received 4 November 2005, in final form 13 January 2006
Published 15 February 2006
Online at stacks.iop.org/JPhysA/39/2177

Abstract
We analyse the renormalizability of the sine-Gordon model using the two-point causal Green function. We show that all divergences can be removed by the renormalization of the dimensional coupling constant using the renormalization constant Z_1, calculated in Faber and Ivanov (2003 *J. Phys. A: Math. Gen.* **36** 7839) within the path-integral approach. We calculate the Gell-Mann–Low function and solve the Callan–Symanzik equation for the two-point Green function. We analyse the renormalizability of Gaussian fluctuations around a soliton. We show that Gaussian fluctuations around a soliton solution are renormalized like quantum fluctuations around the trivial vacuum and do not introduce any singularity to the sine-Gordon model at $\beta^2 = 8\pi$. We calculate the correction to the soliton mass, caused by Gaussian fluctuations around a soliton, within the discretization procedure for various boundary conditions and find complete agreement with our result, obtained in continuous space–time.

PACS numbers: 11.10.Ef, 11.10.Gh, 11.10.Hi, 11.10.Kk

1. Introduction

We describe the sine-Gordon model by the Lagrangian [1, 2]

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0(\Lambda^2)}{\beta^2}(\cos\beta\vartheta(x) - 1), \tag{1.1}$$

where the field $\vartheta(x)$ and the coupling constant β are unrenormalizable, $\alpha_0(\Lambda^2)$ is a dimensional *bare* coupling constant and Λ is an ultra-violet cut-off. As has been shown in [2] the

[1] Permanent address: Department of Nuclear Physics, State Polytechnic University, 195251 St Petersburg, Russia.

coupling constant $\alpha_0(\Lambda^2)$ is multiplicatively renormalizable, and the renormalized Lagrangian reads [2]

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1) + (Z_1 - 1)\frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1)$$
$$= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + Z_1\frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1), \quad (1.2)$$

where $Z_1 = Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)$ is the renormalization constant [2–5] depending on the normalization scale M. The renormalization constant relates the renormalized coupling constant $\alpha_r(M^2)$, depending on the normalization scale M, to the *bare* coupling constant $\alpha_0(\Lambda^2)$ [2–5]

$$\alpha_r(M^2) = Z_1^{-1}(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)\alpha_0(\Lambda^2). \quad (1.3)$$

As has been found in [2] the renormalization constant $Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda)$ is equal to

$$Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda^2) = \left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi}. \quad (1.4)$$

This result is valid to all orders of perturbation theory developed relative to the coupling constant β^2 and $\alpha_0(\Lambda^2)$ [2]. Since the normalization constant does not depend on $\alpha_r(M^2)$, we write below $Z_1 = Z_1(\beta^2, M^2; \Lambda^2)$.

For the analysis of the renormalizability of the sine-Gordon model with respect to quantum fluctuations around the trivial vacuum, we expand the Lagrangian (1.2) in powers of $\vartheta(x)$. This gives

$$\mathcal{L}(x) = \tfrac{1}{2}[\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \alpha_r(M^2)\vartheta^2(x)] + \mathcal{L}_{\rm int}(x), \quad (1.5)$$

where $\mathcal{L}_{\rm int}(x)$ describes the self-interactions of the sine-Gordon field

$$\mathcal{L}_{\rm int}(x) = \alpha_r(M^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x) + (Z_1 - 1)\alpha_r(M^2)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x). \quad (1.6)$$

It is seen that the coupling constant $\alpha_r(M^2)$ has the meaning of a squared mass of free quanta of the sine-Gordon field $\vartheta(x)$. The causal two-point Green function of free sine-Gordon quanta with mass $\alpha_r(M^2)$ is defined by

$$-{\rm i}\Delta_F(x; \alpha_r(M^2)) = \langle 0|{\rm T}(\vartheta(x)\vartheta(0))|0\rangle = \int\frac{{\rm d}^2k}{(2\pi)^2{\rm i}}\frac{{\rm e}^{-{\rm i}k\cdot x}}{\alpha_r(M^2) - k^2 - {\rm i}0}. \quad (1.7)$$

At $x = 0$ the Green function $-{\rm i}\Delta_F(0; \alpha_r(M^2))$ is equal to [2]

$$-{\rm i}\Delta_F(0; \alpha_r(M^2)) = \frac{1}{4\pi}\ln\left[\frac{\Lambda^2}{\alpha_r(M^2)}\right], \quad (1.8)$$

where Λ is a cut-off in the Euclidean two-dimensional momentum space [2].

As usual the generic analysis of the renormalizability of a quantum field theory is carried out in the form of *power counting*, the concept of the superficial degree $\omega(G)$ of divergence of momentum integrals of the Feynman diagram G based on dimensional considerations [3–5]. Following the standard procedure [3–5] one can show that

$$\omega(G) = 2 - 2\sum_{\{n\}}V_{2n}, \quad (1.9)$$

where V_{2n} is the number of $2n$-vertices of the self-interaction term $\vartheta^{2n}(x)$ of the sine-Gordon field $\vartheta(x)$. Relation (1.9) testifies the complete renormalizability of the sine-Gordon model.

The main aim of this paper is to show that the sine-Gordon model is well defined not only for $\beta^2 < 8\pi$ but for $0 \leqslant \beta^2 < \infty$. An important application of this result is the fractional quantum Hall effect (the FQHE) [6, 7]. Indeed, as has been pointed out in [6, 7] the FQHE is defined by the edge tunnelling of quasi-particles and electrons. In the bosonized form the Hamiltonian of the interaction of quasi-particles and electrons has the form of the sine-Gordon interaction [6]

$$\mathcal{H}_{\rm int}(x) = -\frac{\alpha}{\beta^2}\cos\beta\vartheta(x). \tag{1.10}$$

The parameter β^2 is defined by [6]

$$\beta^2 = \begin{cases} 4\pi\nu & \text{for tunnelling of quasi-particles} \\ 4\pi/\nu & \text{for tunnelling of electrons} \end{cases} \tag{1.11}$$

where ν is the filling factor [8]. If the coupling constant β^2 obeys the constraint $\beta^2 < 8\pi$, only quasi-particles can be responsible for the FQHE. The participance of electrons in the FQHE is prohibited. However, if there is a possibility for the coupling constant β^2 to be greater than 8π, i.e. $\beta^2 > 8\pi$, the participance of electrons in the FQHE cannot be suppressed. This opens new possibilities for the dynamics of the FQHE.

The paper is organized as follows. In section 2, we investigate the renormalizability of the two-point Green function of the sine-Gordon field. Quantum fluctuations are calculated relative to the trivial vacuum up to second order in $\alpha_r(M^2)$ and to all orders in β^2. We show that after renormalization of the two-point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2 all higher order corrections in $\alpha_r(M^2)$ and all orders in β^2 can be expressed in terms of $\alpha_{\rm ph}$, the physical dimensional coupling constant independent of the normalization scale M. We derive the effective Lagrangian of the sine-Gordon model, taking into account quantum fluctuations to second order in $\alpha_r(M^2)$ and to all orders in β^2. We show that all divergences can be removed by the renormalization constant (1.4). The first finite correction to $\alpha_{\rm ph}$ is of order $O(\alpha_{\rm ph}\beta^6)$ only. It appears to the second order in $\alpha_r(M^2)$ and the third order in β^2. In section 3, we analyse the renormalizability of the sine-Gordon model within the renormalization group approach. We use the Callan–Symanzik equation for the derivation of the total two-point Green function of the sine-Gordon field in momentum representation. We show that the two-point Green function depends on the running coupling constant $\alpha_r(p^2) = \alpha_{\rm ph}(p^2/\alpha_{\rm ph})^{\tilde{\beta}^2/8\pi}$, where $\tilde{\beta}^2 = \beta^2/(1+\beta^2/8\pi) < 1$ for all β^2. In section 4, we investigate the renormalizability of the sine-Gordon model with respect to Gaussian fluctuations around a soliton solution. We show that Gaussian fluctuations around a soliton solution lead to the same renormalized Lagrangian of the sine-Gordon model as quantum fluctuations around the trivial vacuum taken into account to first order in $\alpha_r(M^2)$ and β^2. In section 5, we discuss the correction to the soliton mass induced by quantum fluctuations. We show that Gaussian fluctuations around a soliton solution reproduce the same correction as the quantum fluctuations around the trivial vacuum, calculated to first orders in $\alpha_r(M^2)$ and β^2. This correction does not contain the finite quantum correction obtained by Dashen *et al* [9, 10] (see also [11]). In section 6, we discuss the calculation of the correction to the soliton mass ΔM_s, induced by Gaussian fluctuations, within a discretization procedure for various boundary conditions. We show that the result of the calculation of ΔM_s does not depend on the boundary conditions and agrees fully with that obtained in continuous space–time. In section 7, we analyse the renormalizability of the causal two-point Green function in the massive sine-Gordon model and compare our results with those obtained by Amit *et al* [12]. In the conclusion, we summarize the obtained results and discuss them.

2. Renormalization of causal two-point Green function

The causal two-point Green function of the sine-Gordon field is defined by

$$-\mathrm{i}\Delta(x) = \frac{1}{\mathrm{i}}\frac{\delta}{\delta J(x)}\frac{1}{\mathrm{i}}\frac{\delta}{\delta J(0)}Z[J]_{J=0}, \tag{2.1}$$

where $Z[J]$ is a generating functional of Green functions

$$\begin{aligned} Z[J] &= \int \mathcal{D}\vartheta \exp\left\{\mathrm{i}\int \mathrm{d}^2y[\mathcal{L}(y) + \vartheta(y)J(y)]\right\} \\ &= \int \mathcal{D}\vartheta \exp\left\{\mathrm{i}\int \mathrm{d}^2y\left[\frac{1}{2}(\partial_\mu\vartheta(y)\partial^\mu\vartheta(y) - \alpha_r(M^2)\vartheta^2(y)) + \mathcal{L}_{\mathrm{int}}(y) + \vartheta(y)J(y)\right]\right\}, \end{aligned} \tag{2.2}$$

normalized by $Z[0] = 1$, $J(x)$ is the external source of the sine-Gordon field $\vartheta(x)$.

Substituting (2.2) into (2.1) we get

$$\begin{aligned} -\mathrm{i}\Delta(x) = &\int \mathcal{D}\vartheta\,\vartheta(x)\vartheta(0) \exp\left\{\mathrm{i}\int \mathrm{d}^2y\mathcal{L}_{\mathrm{int}}(y)\right\} \\ &\times \exp\left\{\frac{\mathrm{i}}{2}\int \mathrm{d}^2y[\partial_\mu\vartheta(y)\partial^\mu\vartheta(y) - \alpha_r(M^2)\vartheta^2(y)]\right\}, \end{aligned} \tag{2.3}$$

where $\mathcal{L}_{\mathrm{int}}(y)$ is given by (1.6). The rhs of (2.3) can be rewritten in the form of a vacuum expectation value of a time-ordered product [13]

$$-\mathrm{i}\Delta(x) = \langle 0|\mathrm{T}\left(\vartheta(x)\vartheta(0)\exp\left\{\mathrm{i}\int \mathrm{d}^2y\mathcal{L}_{\mathrm{int}}(y)\right\}\right)|0\rangle_c, \tag{2.4}$$

where the index c means the *connected* part, $\vartheta(x)$ is the free sine-Gordon field operator with mass $\alpha_r(M^2)$ and the causal two-point Green function $\Delta_F(x, \alpha_r(M^2))$ defined by (1.7).

In the momentum representation, the two-point Green function (2.4) reads

$$\begin{aligned} -\mathrm{i}\tilde{\Delta}(p) &= -\mathrm{i}\int \mathrm{d}^2x\, \mathrm{e}^{+\mathrm{i}p\cdot x}\Delta(x) \\ &= \int \mathrm{d}^2x\, \mathrm{e}^{+\mathrm{i}p\cdot x}\langle 0|\mathrm{T}\left(\vartheta(x)\vartheta(0)\exp\left\{\mathrm{i}\int \mathrm{d}^2y\mathcal{L}_{\mathrm{int}}(y)\right\}\right)|0\rangle_c. \end{aligned} \tag{2.5}$$

For the analysis of the renormalizability of the sine-Gordon model, we propose to calculate the corrections to the two-point Green function (2.4) (or to (2.5)), induced by quantum fluctuations around the trivial vacuum. Expanding the rhs of equation (2.4) in powers of $\alpha_r(M^2)$ and β^2 we determine

$$-\mathrm{i}\Delta(x) = \sum_{m=0}^{\infty}(-\mathrm{i})\Delta^{(m)}(x, \alpha_r(M^2)), \tag{2.6}$$

where $(-\mathrm{i})\Delta^{(m)}(x, \alpha_r(M^2))$ is defined by

$$-\mathrm{i}\Delta^{(m)}(x, \alpha_r(M^2)) = \frac{i^m}{m!}\int\cdots\int \mathrm{d}^2y_1\cdots\mathrm{d}^2y_m\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\mathrm{int}}(y_1)\cdots\mathcal{L}_{\mathrm{int}}(y_m))|0\rangle_c. \tag{2.7}$$

The Green function $(-\mathrm{i})\Delta^{(0)}(x, \alpha_r(M^2))$ coincides with the Green function (1.7) of the free sine-Gordon field.

In the momentum representation, the correction to the two-point Green function $(-\mathrm{i})\Delta^{(m)}(x, \alpha_r(M^2))$ can be written as

$$-\mathrm{i}\tilde{\Delta}^{(m)}(p, \alpha_r(M^2)) = \int \mathrm{d}^2x\, \mathrm{e}^{+\mathrm{i}p\cdot x}(-\mathrm{i})\Delta^{(m)}(x, \alpha_r(M^2)) = \frac{\mathrm{i}^m}{m!}\int \mathrm{d}^2x\, \mathrm{e}^{+\mathrm{i}p\cdot x}$$
$$\times \int \cdots \int \mathrm{d}^2y_1 \cdots \mathrm{d}^2y_m \langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\mathrm{int}}(y_1)\cdots\mathcal{L}_{\mathrm{int}}(y_m))|0\rangle_c. \quad (2.8)$$

Now let us proceed to the analysis of the perturbative corrections to the two-point Green function.

2.1. Two-point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2

The correction to the two-point Green function to first order in $\alpha_r(M^2)$ and to all orders in β^2 is defined by

$$-\mathrm{i}\Delta^{(1)}(x, \alpha_r(M^2)) = \mathrm{i}\int \mathrm{d}^2y_1 \langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\mathrm{int}}(y_1))|0\rangle_c$$
$$= \mathrm{i}\alpha_r(M^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\int \mathrm{d}^2y \langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\vartheta^{2n}(y))|0\rangle_c$$
$$+ \mathrm{i}\alpha_r(M^2)(Z_1 - 1)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\int \mathrm{d}^2y \langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\vartheta^{2n}(y))|0\rangle_c. \quad (2.9)$$

Making all contractions we arrive at the expression

$$-\mathrm{i}\Delta^{(1)}(x, \alpha_r(M^2)) = \mathrm{i}\alpha_r(M^2)\left[1 - Z_1 \exp\left\{\frac{1}{2}\beta^2 \mathrm{i}\Delta_F(0, \alpha_r(M^2))\right\}\right]$$
$$\times \int \mathrm{d}^2y[-\mathrm{i}\Delta_F(x - y, \alpha_r(M^2))][-\mathrm{i}\Delta_F(-y, \alpha_r(M^2))]. \quad (2.10)$$

Using the normalization constant Z_1, given by (1.4), and definition (1.8) of the two-point Green function we remove the cut-off Λ

$$-\mathrm{i}\Delta^{(1)}(x, \alpha_r(M^2)) = \mathrm{i}\alpha_r(M^2)\left[1 - \left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]$$
$$\times \int \mathrm{d}^2y[-\mathrm{i}\Delta_F(x - y, \alpha_r(M^2))][-\mathrm{i}\Delta_F(-y, \alpha_r(M^2))]. \quad (2.11)$$

Thus, the renormalized causal two-point Green function of the sine-Gordon field, defined to first order in $\alpha_r(M^2)$ and to all orders in β^2, is given by

$$-\mathrm{i}\Delta(x) = -\mathrm{i}\Delta_F(x, \alpha_r(M^2)) + \mathrm{i}\alpha_r(M^2)\left[1 - \left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]$$
$$\times \int \mathrm{d}^2y[-\mathrm{i}\Delta_F(x - y, \alpha_r(M^2))][-\mathrm{i}\Delta_F(-y, \alpha_r(M^2))]. \quad (2.12)$$

In the momentum representation, the two-point Green function (2.12) reads

$$-\mathrm{i}\tilde{\Delta}(p) = \frac{(-\mathrm{i})}{\alpha_r(M^2) - p^2} + \frac{(-\mathrm{i})}{\alpha_r(M^2) - p^2}\mathrm{i}\alpha_r(M^2)\left[1 - \left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]\frac{(-\mathrm{i})}{\alpha_r(M^2) - p^2}. \quad (2.13)$$

The second term defines the correction to the mass of the sine-Gordon field

$$\delta\alpha_r(M^2) = -\alpha_r(M^2)\left[1 - \left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]. \tag{2.14}$$

Thus, the two-point Green function, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2, is equal to

$$-\mathrm{i}\tilde{\Delta}(p) = \frac{(-\mathrm{i})}{\alpha_r(M^2) + \delta\alpha_r(M^2) - p^2} = \frac{(-\mathrm{i})}{\alpha_{\mathrm{ph}} - p^2}, \tag{2.15}$$

where α_{ph} is determined by

$$\alpha_{\mathrm{ph}} = \alpha_r(M^2) + \delta\alpha_r(M^2) = \alpha_r(M^2)\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}. \tag{2.16}$$

This gives also $\alpha_r(M^2)$ in term of M and α_{ph}

$$\alpha_r(M^2) = \alpha_{\mathrm{ph}}\left(\frac{M^2}{\alpha_{\mathrm{ph}}}\right)^{\tilde{\beta}^2/8\pi}, \qquad \tilde{\beta}^2 = \frac{\beta^2}{1 + \dfrac{\beta^2}{8\pi}}. \tag{2.17}$$

The Green function (2.15) can be obtained to leading order in β^2 from the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}(x) &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_r(M^2)}{\beta^2}\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}(\cos\beta\vartheta(x) - 1)\\ &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\mathrm{ph}}}{\beta^2}(\cos\beta\vartheta(x) - 1).\end{aligned} \tag{2.18}$$

We argue that higher order corrections to the two-point Green function in $\alpha_r(M^2)$ and to all orders in β^2 should depend on the physical coupling constant α_{ph} only

$$\begin{aligned}-\mathrm{i}\Delta^{(m)}(x, \alpha_r(M^2)) &= \frac{\mathrm{i}^m}{m!}\int\cdots\int \mathrm{d}^2y_1\cdots\mathrm{d}^2y_m\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\mathrm{int}}(y_1)\cdots\mathcal{L}_{\mathrm{int}}(y_m))|0\rangle_c\\ &= -\mathrm{i}\Delta^{(m)}(x, \alpha_{\mathrm{ph}}) \qquad (\text{for } m \geqslant 2).\end{aligned} \tag{2.19}$$

In order to prove this assertion it is sufficient to analyse the renormalization of the causal two-point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2.

2.2. Two-point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2

The correction to the two-point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2 is defined by

$$-\mathrm{i}\Delta^{(2)}(x, \alpha_r(M^2)) = -\frac{1}{2}\iint \mathrm{d}^2y_1\mathrm{d}^2y_2\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\mathrm{int}}(y_1)\mathcal{L}_{\mathrm{int}}(y_2))|0\rangle_c. \tag{2.20}$$

After the extraction of the contributions corresponding to the second order of the expansion of the geometric series, the non-trivial contribution is

$$\begin{aligned}-\mathrm{i}\Delta^{(2)}(x, \alpha_r(M^2)) &= \left[\alpha_r(M^2)Z_1\exp\left\{\frac{\beta^2}{2}\mathrm{i}\Delta_F(0, \alpha_r(M^2))\right\}\right]^2\frac{1}{\beta^2}\iint \mathrm{d}^2y_1\,\mathrm{d}^2y_2\\ &\quad\times[-\mathrm{i}\Delta_F(x - y_1, \alpha_r(M^2))]\Big(\cosh[-\beta^2\mathrm{i}\Delta_F(y_1 - y_2, \alpha_r(M^2))] - 1\\ &\quad - \frac{1}{2}\beta^4[-\mathrm{i}\Delta_F(y_1 - y_2, \alpha_r(M^2))]^2\Big)[-\mathrm{i}\Delta_F(y_1, \alpha_r(M^2))]\end{aligned}$$

$$- \left[\alpha_r(M^2) Z_1 \exp\left\{\frac{\beta^2}{2}\mathrm{i}\Delta_F(0,\alpha_r(M^2))\right\}\right]^2$$
$$\times \frac{1}{\beta^2}\iint \mathrm{d}^2y_1\,\mathrm{d}^2y_2[-\mathrm{i}\Delta_F(x-y_1,\alpha_r(M^2))](\sinh[-\beta^2\mathrm{i}\Delta_F(y_1-y_2,\alpha_r(M^2))]$$
$$-\beta^2[-\mathrm{i}\Delta_F(y_1-y_2,\alpha_r(M^2))])[-\mathrm{i}\Delta_F(y_2,\alpha_r(M^2))]. \tag{2.21}$$

Since the coupling constant $\alpha_r(M^2)$ can be replaced everywhere by $\alpha_{\rm ph}$ by means of the renormalization constant Z_1 (1.4), in the momentum representation the correction to the two-point Green function to second order in $\alpha_{\rm ph}$ takes the form

$$-\mathrm{i}\tilde{\Delta}^{(2)}(p,\alpha_r(M^2)) = \mathrm{i}\alpha_{\rm ph}^2\left[\frac{(-\mathrm{i})}{\alpha_{\rm ph}-p^2}\right]^2$$
$$\times\left\{\sum_{n=2}^{\infty}\frac{\beta^{4n-4}}{(2n-1)!}\int\frac{\mathrm{d}^2q_1}{(2\pi)^2\mathrm{i}}\frac{1}{\alpha_{\rm ph}-q_1^2}\int\frac{\mathrm{d}^2q_2}{(2\pi)^2\mathrm{i}}\frac{1}{\alpha_{\rm ph}-q_2^2}\cdots\right.$$
$$\times\int\frac{\mathrm{d}^2q_{2n-2}}{(2\pi)^2\mathrm{i}}\frac{1}{\alpha_{\rm ph}-q_{2n-2}^2}\frac{1}{\alpha_{\rm ph}-(p-q_1-q_2-\cdots-q_{2n-2})^2}$$
$$-\sum_{n=2}^{\infty}\frac{\beta^{4n-2}}{(2n)!}\int\frac{\mathrm{d}^2q_1}{(2\pi)^2\mathrm{i}}\frac{1}{\alpha_{\rm ph}-q_1^2}\int\frac{\mathrm{d}^2q_2}{(2\pi)^2\mathrm{i}}\frac{1}{\alpha_{\rm ph}-q_2^2}\cdots$$
$$\left.\times\int\frac{\mathrm{d}^2q_{2n-1}}{(2\pi)^2\mathrm{i}}\frac{1}{\alpha_{\rm ph}-q_{2n-1}^2}\frac{1}{\alpha_{\rm ph}-(q_1+q_2+\cdots+q_{2n-1})^2}\right\} = -\mathrm{i}\tilde{\Delta}^{(2)}(p,\alpha_{\rm ph}). \tag{2.22}$$

This proves relation (2.19) to second order in $\alpha_r(M^2)$ and to all orders in β^2. The proof of relation (2.19) to arbitrary orders in $\alpha_r(M^2)$ and β^2 runs in the same way.

The even-order corrections in β^2, i.e. $O\big(\alpha_{\rm ph}^2(\beta^2)^{2n-2}\big)$, introduce a non-trivial momentum dependence of the two-point Green function, whereas the odd-order corrections in β^2, i.e. $O\big(\alpha_{\rm ph}^2(\beta^2)^{2n-1}\big)$, do not depend on the momentum p. They contribute to the effective coupling constant $\alpha_{\rm eff} = \alpha_{\rm ph} f(\beta^2)$, where $f(\beta^2) = O(\beta^6)$.

2.3. *Physical renormalization of the sine-Gordon model*

Thus, using the results obtained above we can formulate a procedure for the renormalization of the sine-Gordon model dealing with physical parameters only. Starting with the Lagrangian (1.1) and making a renormalization at the normalization scale $M^2 = \alpha_{\rm ph}$ we deal with physical parameters only

$$\alpha_{\rm ph} = Z_1^{-1}(\beta^2,\alpha_{\rm ph};\Lambda^2)\alpha_0(\Lambda^2) = \alpha_0(\Lambda^2)\left(\frac{\alpha_{\rm ph}}{\Lambda^2}\right)^{\beta^2/8\pi}. \tag{2.23}$$

The renormalized Lagrangian is defined by

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\rm ph}}{\beta^2}(\cos\beta\vartheta(x)-1) + (Z_1-1)\frac{\alpha_{\rm ph}}{\beta^2}(\cos\beta\vartheta(x)-1) \tag{2.24}$$

with the renormalization constant $Z_1(\beta^2,\alpha_{\rm ph};\Lambda^2) = (\Lambda^2/\alpha_{\rm ph})^{\beta^2/8\pi}$. From relation (2.17) at $M^2 = \alpha_{\rm ph}$ one can obtain that $\alpha_r(\alpha_{\rm ph}) = \alpha_{\rm ph}$. The calculation of perturbative corrections to the two-point Green function of the sine-Gordon model shows that the first-order correction in $\alpha_{\rm ph}$ vanishes in accordance with equation (2.14). Non-trivial corrections appear only to second and higher orders in $\alpha_{\rm ph}$.

One can also show that the results obtained within the physical renormalization of the sine-Gordon model can be fully reproduced by using the normal-ordered Lagrangian

$$\mathcal{L}(x) = \frac{1}{2} : \partial_\mu \vartheta(x) \partial^\mu \vartheta(x) : + \frac{\alpha_{\rm ph}}{\beta^2} : (\cos \beta \vartheta(x) - 1) : . \tag{2.25}$$

In this case all corrections to the two-point Green function are expressed in terms of $\alpha_{\rm ph}$ and finite.

3. Renormalization group analysis

In this section, we discuss the renormalization group approach [3–5] to the renormalization of the sine-Gordon model. We apply the Callan–Symanzik equation to the analysis of the Fourier transform of the two-point Green function of the sine-Gordon field.

The Callan–Symanzik equation for the Fourier transform of the two-point Green function of the sine-Gordon field (2.5), which we denote below as $-{\rm i}\tilde{\Delta}(p; \alpha_r(M^2), \beta^2)$, is equal to [3]

$$\left[-p \cdot \frac{\partial}{\partial p} + \beta(\alpha_r(M^2), \beta^2) \frac{\partial}{\partial \alpha_r(M^2)} - 2 \right] \tilde{\Delta}(p; \alpha_r(M^2), \beta^2) = F(0, p; \alpha_r(M^2), \beta^2), \tag{3.1}$$

where $\beta(\alpha_r(M^2), \beta^2)$ is the Gell-Mann–Low function

$$M \frac{\partial \alpha_r(M^2)}{\partial M} = \beta(\alpha_r(M^2), \beta^2). \tag{3.2}$$

The term $\gamma(\alpha_r(M^2), \beta^2)$, describing an anomalous dimension of the sine-Gordon field, is equal to zero [3]. The rhs of (3.1) is defined by

$$F(0, p; \alpha_r(M^2), \beta^2) = \iint {\rm d}^2x \, {\rm d}^2y \, {\rm e}^{+{\rm i}p \cdot x} \langle 0|{\rm T}\big(\Theta^\mu_\mu(y)\vartheta(x)\vartheta(0) \, {\rm e}^{{\rm i}\int {\rm d}^2y \mathcal{L}_{\rm int}(y)}\big)|0\rangle_c, \tag{3.3}$$

where $\mathcal{L}_{\rm int}(y)$ is equal to [2]

$$\mathcal{L}_{\rm int}(y) = \frac{\alpha_0}{\beta^2}(\cos \beta \vartheta(x) - 1). \tag{3.4}$$

Then, $\Theta^\mu_\mu(y)$ is the trace of the energy–momentum tensor $\Theta_{\mu\nu}(x)$. For a (pseudo)scalar field $\vartheta(x)$, described by the Lagrangian $\mathcal{L}(x)$, it is defined by [3]

$$\Theta_{\mu\nu}(x) = \frac{\partial \mathcal{L}(x)}{\partial^\mu \vartheta(x)} \partial_\nu \vartheta(x) - g_{\mu\nu} \mathcal{L}(x). \tag{3.5}$$

Using the Lagrange equation of motion one can show that

$$\partial^\mu \Theta_{\mu\nu}(x) = 0. \tag{3.6}$$

For the sine-Gordon model, the energy–momentum tensor $\Theta_{\mu\nu}(x)$ reads

$$\Theta_{\mu\nu}(x) = \partial_\mu \vartheta(x) \partial_\nu \vartheta(x) - g_{\mu\nu} \left[\frac{1}{2} \partial_\lambda \vartheta(x) \partial^\lambda \vartheta(x) + \frac{\alpha_0}{\beta^2} (\cos \beta \vartheta(x) - 1) \right]. \tag{3.7}$$

The trace of the energy–momentum tensor $\Theta_{\mu\nu}(x)$ is equal to

$$\Theta^\mu_\mu(x) = -\frac{2\alpha_0}{\beta^2}(\cos \beta \vartheta(x) - 1) = 2V[\vartheta(x)], \tag{3.8}$$

where $V[\vartheta(x)]$ is the potential density functional of the sine-Gordon field $\vartheta(x)$.

Since the trace of the energy–momentum tensor is proportional to the potential energy density, the Fourier transform $F(0, p; \alpha_r(M^2), \beta^2)$ can be related to the two-point Green function as

$$F(0, p; \alpha_r(M^2), \beta^2) = 2\alpha_r(M^2)\frac{\partial}{\partial \alpha_r(M^2)}\tilde{\Delta}(p; \alpha_r(M^2), \beta^2), \tag{3.9}$$

where we have used the definition of the trace $\Theta^{\mu}_{\mu}(y)$ of the energy–momentum tensor equation (3.8) and the relation $\alpha_0 = \alpha_r(M^2)Z_1(\beta^2, M^2; \Lambda^2)$.

Substituting (3.9) into (3.1) we arrive at the Callan–Symanzik equation for the Fourier transform of the two-point Green function of the sine-Gordon field

$$\left[-p^2\frac{\partial}{\partial p^2} + \left(\frac{1}{2}\beta(\alpha_r(M^2), \beta^2) - \alpha_r(M^2)\right)\frac{\partial}{\partial \alpha_r(M^2)} - 1\right]\tilde{\Delta}(p^2; \alpha_r(M^2), \beta^2) = 0, \tag{3.10}$$

where we have taken into account that $\tilde{\Delta}(p; \alpha_r(M^2), \beta^2)$ should depend on p^2 due to Lorentz covariance.

For the solution of (3.10) we have to determine the Gell-Mann–Low function (3.2). For the coupling constant $\alpha_r(M^2)$, defined by (2.17), the Gell-Mann–Low function is

$$\beta(\alpha_r(M^2), \beta^2) = \frac{\tilde{\beta}^2}{4\pi}\alpha_r(M^2), \tag{3.11}$$

where $\tilde{\beta}^2 = \beta^2/(1 + \beta^2/8\pi)$ (2.17). This gives the Callan–Symanzik equation

$$\left[p^2\frac{\partial}{\partial p^2} + \left(1 - \frac{\tilde{\beta}^2}{8\pi}\right)\alpha_r(M^2)\frac{\partial}{\partial \alpha_r(M^2)} + 1\right]\tilde{\Delta}(p^2; \alpha_r(M^2), \beta^2) = 0. \tag{3.12}$$

Setting $\tilde{\Delta}(p^2; \alpha_r(M^2), \beta^2) = D(p^2; \alpha_r(M^2), \beta^2)/p^2$ we get

$$\left[p^2\frac{\partial}{\partial p^2} + \left(1 - \frac{\tilde{\beta}^2}{8\pi}\right)\alpha_r(M^2)\frac{\partial}{\partial \alpha_r(M^2)}\right]D(p^2; \alpha_r(M^2), \beta^2) = 0. \tag{3.13}$$

Due to dimensional consideration the function $D(p^2; \alpha_r(M^2), \beta^2)$ should be dimensionless, depending on the dimensionless variables $\tilde{p}^2 = p^2/M^2$ and $\tilde{\alpha} = \alpha_r(M^2)/M^2$, where M is a normalization scale. This gives

$$\left[\tilde{p}^2\frac{\partial}{\partial \tilde{p}^2} + \left(1 - \frac{\tilde{\beta}^2}{8\pi}\right)\tilde{\alpha}\frac{\partial}{\partial \tilde{\alpha}}\right]D(\tilde{p}^2; \tilde{\alpha}, \beta^2) = 0. \tag{3.14}$$

According to the general theory of partial differential equations of first order [15], the solution of (3.14) is an arbitrary function of the integration constant

$$C = \frac{\tilde{\alpha}}{\tilde{p}^2}(\tilde{p}^2)^{\tilde{\beta}^2/8\pi}, \tag{3.15}$$

which is the solution of the characteristic differential equation

$$\left(1 - \frac{\tilde{\beta}^2}{8\pi}\right)\frac{\mathrm{d}\tilde{p}^2}{\tilde{p}^2} = \frac{\mathrm{d}\tilde{\alpha}}{\tilde{\alpha}}. \tag{3.16}$$

Hence, the Fourier transform of the two-point Green function of the sine-Gordon field is equal to

$$\tilde{\Delta}(p^2; \alpha_r(M^2), \beta^2) = \frac{1}{p^2}D\left[\frac{\alpha_r(M^2)}{p^2}\left(\frac{p^2}{M^2}\right)^{\tilde{\beta}^2/8\pi}\right]. \tag{3.17}$$

The argument of the D-function can be expressed in terms of the running coupling constant $\alpha_r(p^2)$

$$\alpha_r(p^2) = \alpha_r(M^2)\left(\frac{p^2}{M^2}\right)^{\tilde{\beta}^2/8\pi} = \alpha_{\rm ph}\left(\frac{M^2}{\alpha_{\rm ph}}\right)^{\tilde{\beta}^2/8\pi}\left(\frac{p^2}{M^2}\right)^{\tilde{\beta}^2/8\pi} = \alpha_{\rm ph}\left(\frac{p^2}{\alpha_{\rm ph}}\right)^{\tilde{\beta}^2/8\pi}. \tag{3.18}$$

The solution of the Callan–Symanzik equation for the Fourier transform of the two-point Green function of the sine-Gordon field is

$$\tilde{\Delta}(p^2; \alpha_{\rm ph}, \beta^2) = \frac{1}{p^2} D\left[\frac{\alpha_r(p^2)}{p^2}\right]. \tag{3.19}$$

This proves that the total renormalized two-point Green function of the sine-Gordon field depends on the physical coupling constant $\alpha_{\rm ph}$ only.

4. Renormalization of Gaussian fluctuations around solitons

We apply the renormalization procedure expounded above to the calculation of the contribution of quantum fluctuations around a soliton solution. We start with the partition function

$$\begin{aligned} Z_{\rm SG} &= \int \mathcal{D}\vartheta \exp\left\{{\rm i}\int {\rm d}^2x\left[\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0}{\beta^2}(\cos\beta\vartheta(x) - 1)\right]\right\} \\ &= \int \mathcal{D}\vartheta \exp\left\{{\rm i}\int {\rm d}^2x \mathcal{L}[\vartheta(x)]\right\}. \end{aligned} \tag{4.1}$$

Following Dashen *et al* [9, 10] (see also [16]) we treat the quantum fluctuations of the sine-Gordon field $\vartheta(x)$ around the classical solution $\vartheta(x) = \vartheta_{c\ell}(x) + \varphi(x)$, where $\varphi(x)$ is the field fluctuating around $\vartheta_{c\ell}(x)$, the single soliton solution of the classical equation of motion

$$\Box\,\vartheta_{c\ell}(x) + \frac{\alpha_0}{\beta}\sin\beta\vartheta_{c\ell}(x) = 0 \tag{4.2}$$

equal to [9, 11, 16]

$$\vartheta_{c\ell}(x) = \frac{4}{\beta}\arctan(\exp(\sqrt{\alpha_0}\gamma(x^1 - ux^0)) = \frac{4}{\beta}\arctan(\exp(\sqrt{\alpha_0}\sigma)), \tag{4.3}$$

where u is the velocity of the soliton, $\sigma = \gamma(x^1 - ux^0)$ and $\gamma = 1/\sqrt{1-u^2}$.[2]

Substituting $\vartheta(x) = \vartheta_{c\ell}(x) + \varphi(x)$ into the exponent of the integrand of (4.1), using the equation of motion for the soliton solution $\vartheta_{c\ell}(x)$, and dealing with Gaussian fluctuations only [9, 10], we transcribe the partition function (4.1) into the form

$$\begin{aligned} Z_{\rm SG} = \exp\left\{{\rm i}\int {\rm d}^2x\mathcal{L}[\vartheta_{c\ell}(x)]\right\}\int \mathcal{D}\varphi \exp\left\{-{\rm i}\frac{1}{2}\int {\rm d}\tau\,{\rm d}\sigma\varphi(\tau,\sigma)\right. \\ \left.\times\left[\Box + \alpha_0 - \frac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\right]\varphi(\tau,\sigma)\right\}. \end{aligned} \tag{4.4}$$

It is seen that $\sqrt{\alpha_0}$ has the distinct meaning of the mass of the quanta of the Klein–Gordon field $\varphi(\tau, \sigma)$ coupled to an external force described by a scalar potential[3].

Integrating over the fluctuating field $\varphi(\tau, \sigma)$, we transcribe the rhs of (4.4) into the form

$$Z_{\rm SG} = \exp\left\{{\rm i}\int {\rm d}^2x\mathcal{L}[\vartheta_{c\ell}(x)] + {\rm i}\delta\mathcal{S}[\vartheta_{c\ell}]\right\}. \tag{4.5}$$

[2] In analogy with the 'spatial' variable σ we can define the 'time' variable for the soliton moving with velocity u as $\tau = \gamma(x^0 - ux^1)$. In variables (τ, σ) an infinitesimal element of the two-dimensional volume ${\rm d}^2x$ is equal to ${\rm d}^2x = {\rm d}\tau{\rm d}\sigma$ and the d'Alembert operator $\Box$ is defined by $\Box = \partial^2/\partial\tau^2 - \partial^2/\partial\sigma^2$.

[3] The parameter α_0 should enter with the imaginary correction $\alpha_0 \to \alpha_0 - {\rm i}0$. This is required by the convergence of the path integral [18].

We have denoted

$$\exp\{i\delta\mathcal{S}[\vartheta_{c\ell}]\} = \exp\{i\int d^2x\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x)]\} = \sqrt{\frac{{\rm Det}(\Box + \alpha_0)}{{\rm Det}\left(\Box + \alpha_0 - \dfrac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\right)}}$$

$$= \exp\left\{-\frac{1}{2}\sum_n \ln\lambda_n + \frac{1}{2}\int d^2x\int \frac{d^2p}{(2\pi)^2}\ln(\alpha_0 - p^2)\right\}, \qquad (4.6)$$

where p is a $(1+1)$-dimensional momentum. The second term in the exponent corresponds to the subtraction of the vacuum contribution. The effective action, caused by fluctuations around a soliton solution, is defined by

$$\delta\mathcal{S}[\vartheta] = \int d^2x\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x)] = i\frac{1}{2}\sum_n \ln\lambda_n + \frac{1}{2}\int d^2x\int \frac{d^2p}{(2\pi)^2 i}\ln(\alpha_0 - p^2), \qquad (4.7)$$

where λ_n are the eigenvalues of the equation

$$\left(\Box + \alpha_0 - \frac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\right)\varphi_n(\tau,\sigma) = \lambda_n\varphi_n(\tau,\sigma) \qquad (4.8)$$

and $\varphi_n(\tau,\sigma)$ are eigenfunctions. The quantum number n can be both discrete and continuous. This implies that the sum over n in (4.6) should contain both the summation over the discrete values of the quantum number n and integration over the continuous ones.

According to the Fourier method [17], the solution of equation (4.8) should be taken in the form

$$\varphi_n(\tau,\sigma) = e^{-i\omega\tau}\psi_n(\sigma), \qquad (4.9)$$

where $-\infty \leqslant \omega \leqslant +\infty$ and $\psi_n(\sigma)$ is a complex function[4].

Substituting (4.9) into (4.8) we get

$$\left(\frac{d^2}{d\sigma^2} + k^2 + \frac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}\sigma)}\right)\psi_n(\tau,\sigma) = 0, \qquad (4.10)$$

where we have denoted

$$k^2 = \lambda_n + \omega^2 - \alpha_0. \qquad (4.11)$$

This defines eigenvalues λ_n as functions of ω and k

$$\lambda_n = \alpha_0 - \omega^2 + k^2. \qquad (4.12)$$

The parameter k has the meaning of a spatial momentum $-\infty < k < +\infty$. The solutions of equation (4.10) are[5]

$$\psi_b(\sigma) = \sqrt{\frac{\sqrt{\alpha_0}}{2}}\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)}, \qquad \psi_k(\sigma) = \frac{i}{\sqrt{2\pi}}\frac{-ik + \sqrt{\alpha_0}\tanh(\sqrt{\alpha_0}\sigma)}{\sqrt{k^2+\alpha_0}}\,e^{+ik\sigma}, \qquad (4.13)$$

where the eigenfunction $\psi_b(\sigma)$ has eigenvalues $\lambda_n = -\omega^2$ and the eigenfunctions $\psi_k(\sigma)$ have eigenvalues $\lambda_n = \alpha_0 - \omega^2 + k^2$. In the asymptotic region $\sigma \to \infty$ the function $\psi_k(\sigma)$ behaves as

$$\psi_k(\sigma) \to \frac{1}{\sqrt{2\pi}}\,e^{+ik\sigma + i\frac{1}{2}\delta(k)}, \qquad (4.14)$$

[4] Since $\varphi_n(\tau,\sigma)$ is a real field, we have to take the real part of the solution (4.9) only, i.e. $\varphi(\tau,\sigma) = \mathcal{R}e(e^{-i\omega\tau}\psi(\sigma))$. Though without loss of generality, one can also use complex eigenfunctions [10, 11, 16].

[5] The solutions of equation (4.10) are well known [16] (see also [10, 11].

where $\delta(k)$ is a phase shift defined by [16]

$$\delta(k) = 2\arctan\frac{\sqrt{\alpha_0}}{k}. \tag{4.15}$$

The solutions (4.13) satisfy the completeness condition [16]

$$\int_{-\infty}^{+\infty} \mathrm{d}k\,\psi_k^*(\sigma')\psi_k(\sigma) + \psi_b(\sigma')\psi_b(\sigma) = \delta(\sigma' - \sigma). \tag{4.16}$$

The fluctuating field $\varphi(\tau,\sigma)$ is equal to

$$\begin{aligned}\varphi_{\omega b}(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{i}\omega\tau}\psi_b(\sigma) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sqrt{\alpha_0}}{2}}\frac{1}{\cosh(\sqrt{\alpha_0}\sigma)}\,\mathrm{e}^{-\mathrm{i}\omega\tau},\\ \varphi_{\omega k}(\tau,\sigma) &= \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{i}\omega\tau}\psi_k(\sigma) = \frac{\mathrm{i}}{2\pi}\frac{-\mathrm{i}k + \sqrt{\alpha_0}\tanh(\sqrt{\alpha_0}\sigma)}{\sqrt{k^2+\alpha_0}}\,\mathrm{e}^{-\mathrm{i}\omega\tau + ik\sigma}.\end{aligned} \tag{4.17}$$

In terms of the eigenvalues $\lambda_n = -\omega^2$ and $\lambda_n = \alpha_0 - \omega^2 + k^2$ and eigenfunctions (4.17) the effective action $\delta\mathcal{S}[\vartheta_{c\ell}]$ is determined by

$$\begin{aligned}\delta\mathcal{S}[\vartheta_{c\ell}] = &-\frac{1}{2}\int \mathrm{d}^2x \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\mathrm{d}k|\psi_k(x)|^2\ln(\alpha_0 - \omega^2 + k^2)\\ &-\frac{1}{2}\int \mathrm{d}^2x \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}|\psi_b(x)|^2\ln(-\omega^2) + \frac{1}{2}\int \mathrm{d}^2x\int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\ln(\alpha_0 - p^2).\end{aligned} \tag{4.18}$$

Using the explicit expressions for the eigenfunctions $\psi_k(x)$ and $\psi_b(x)$ we reduce the rhs of (4.18) to the form

$$\begin{aligned}\delta\mathcal{S}[\vartheta_{c\ell}] = &-\frac{1}{2}\int \mathrm{d}^2x \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\frac{k^2 + \alpha_0\tanh^2(\sqrt{\alpha_0}\sigma)}{k^2+\alpha_0}\ln(\alpha_0 - \omega^2 + k^2)\\ &-\frac{1}{2}\int \mathrm{d}^2x \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\sqrt{\alpha_0}}{2}\frac{1}{\cosh^2(\sqrt{\alpha_0}\sigma)}\ln(-\omega^2) + \frac{1}{2}\int \mathrm{d}^2x\int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\ln(\alpha_0 - p^2).\end{aligned} \tag{4.19}$$

Introducing the notation

$$\frac{1}{\cosh^2(\sqrt{\alpha_0}\sigma)} = \frac{1}{2}(1 - \cos\beta\vartheta_{c\ell}(x)) = \frac{\beta^2}{2\alpha_0}V[\vartheta_{c\ell}(x)] \tag{4.20}$$

we obtain the effective Lagrangian $\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{c\ell}(x)]$. It is equal to[6]

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{c\ell}(x)] = &-\frac{1}{4}\beta^2V[\vartheta_{c\ell}(x)]\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{k^2+\alpha_0}[\ln(-\omega^2) - \ln(\alpha_0 - \omega^2 + k^2)]\\ &-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\ln(\alpha_0 - \omega^2 + k^2) + \frac{1}{2}\int\frac{\mathrm{d}^2p}{(2\pi)^2\mathrm{i}}\ln(\alpha_0 - p^2).\end{aligned} \tag{4.21}$$

The two last terms cancel each other. This gives

$$\delta\mathcal{L}_{\mathrm{eff}}[\vartheta_{c\ell}(x)] = -\frac{1}{4}\beta^2V[\vartheta_{c\ell}(x)]\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{k^2+\alpha_0}[\ln(-\omega^2) - \ln(\alpha_0 - \omega^2 + k^2)]. \tag{4.22}$$

[6] In the contribution of the mode $\lambda_n = -\omega^2$ we have used the integral representation

$$\frac{1}{2\sqrt{\alpha_0}} = \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{k^2+\alpha_0}.$$

After the integration by parts over ω the effective Lagrangian $\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)]$ is defined by

$$\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)] = \frac{1}{2}\beta^2 V[\vartheta_{\rm c\ell}(x)]\int_{-\infty}^{+\infty}\frac{{\rm d}k}{2\pi}\int_{-\infty}^{+\infty}\frac{{\rm d}\omega}{2\pi i}\frac{1}{\alpha_0-\omega^2+k^2-{\rm i}0}. \tag{4.23}$$

The appearance of the imaginary correction $-{\rm i}0$ is caused by the convergence of the path integral (4.4) [18].

Integrating over ω we reduce the rhs of (4.23) to the form

$$\begin{aligned}\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)] &= \frac{1}{2}\beta^2 V[\vartheta_{\rm c\ell}(x)]\int_{-\infty}^{+\infty}\frac{{\rm d}k}{4\pi}\frac{1}{\sqrt{\alpha_0+k^2}}\\ &= -\frac{\beta^2}{4\sqrt{\alpha_0}}V[\vartheta_{\rm c\ell}(x)]\int_{-\infty}^{+\infty}\frac{{\rm d}k}{4\pi}\sqrt{\alpha_0+k^2}\frac{{\rm d}\delta(k)}{{\rm d}k},\end{aligned} \tag{4.24}$$

where $\delta(k)$ is a phase shift defined in equation (4.15). We discuss this expression in section 7 in connection with the correction to the soliton mass caused by Gaussian fluctuations.

For the analysis of the renormalizability of the sine-Gordon model, the momentum integral in the effective Lagrangian $\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)]$ should be taken in a Lorentz covariant form (4.23) and regularized in a covariant way. Making a Wick rotation $\omega \to {\rm i}\omega$ and passing to Euclidean momentum space, we define the integral over ω and k in (4.23) as [2]

$$\int_{-\infty}^{+\infty}\frac{{\rm d}k}{2\pi}\int_{-\infty}^{+\infty}\frac{{\rm d}\omega}{2\pi {\rm i}}\frac{1}{\alpha_0-\omega^2+k^2-{\rm i}0} = \frac{1}{4\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right), \tag{4.25}$$

where Λ is an Euclidean cut-off [2]. The effective Lagrangian, induced by Gaussian fluctuations around a soliton solution, is equal to

$$\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm c\ell}(x)] = \frac{\alpha_0}{\beta^2}\left[-\frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right)\right](\cos\beta\vartheta_{\rm c\ell}(x)-1). \tag{4.26}$$

The Lagrangian (4.26) has the distinct form of the correction, caused by quantum fluctuations around the trivial vacuum calculated to first orders in $\alpha_0(\Lambda^2)$ and β^2.

The total Lagrangian, accounting for Gaussian fluctuations around the soliton solution, amounts to

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta_{\rm c\ell}(x)\partial^\mu\vartheta_{\rm c\ell}(x) + \frac{\alpha_0}{\beta^2}\left[1-\frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right)\right](\cos\beta\vartheta_{\rm c\ell}(x)-1). \tag{4.27}$$

This coincides with equation (6.7) of [2].

As has been shown in [2], the dependence of the effective Lagrangian (4.27) on the cut-off Λ can be removed by renormalization with the renormalization constant (1.4)

$$\begin{aligned}\alpha_0\left[1-\frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_0}\right)\right] &= \alpha_r(M^2)Z_1\left[1-\frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\right)\right]\\ &= \alpha_r(M^2)\left[1+\frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{M^2}\right)-\frac{\beta^2}{8\pi}\ln\left(\frac{\Lambda^2}{\alpha_r(M^2)}\right)\right]\\ &= \alpha_r(M^2)\left[1+\frac{\beta^2}{8\pi}\ln\left(\frac{\alpha_r(M^2)}{M^2}\right)\right],\end{aligned} \tag{4.28}$$

where we have kept terms of order $O(\beta^2)$ in the β^2-expansion of the renormalization constant (1.4). This gives the effective Lagrangian

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta_{\rm c\ell}(x)\partial^\mu\vartheta_{\rm c\ell}(x) + \frac{\alpha_r(M^2)}{\beta^2}\left[1+\frac{\beta^2}{8\pi}\ln\left(\frac{\alpha_r(M^2)}{M^2}\right)\right](\cos\beta\vartheta_{\rm c\ell}(x)-1). \tag{4.29}$$

We can replace the coupling constant $\alpha_r(M^2)$ by the physical coupling constant $\alpha_{\rm ph}$, related to $\alpha_r(M^2)$ by (2.16)

$$\alpha_{\rm ph} = \alpha_r(M^2)\left[1 + \frac{\beta^2}{8\pi}\ln\left(\frac{\alpha_r(M^2)}{M^2}\right)\right], \tag{4.30}$$

where we have kept terms of order $O(\beta^2)$ only. Substituting (4.30) into (4.29) we get

$$\mathcal{L}^{(r)}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta_{c\ell}(x)\partial^\mu\vartheta_{c\ell}(x) + \frac{\alpha_{\rm ph}}{\beta^2}(\cos\beta\vartheta_{c\ell}(x) - 1). \tag{4.31}$$

The effective Lagrangian (4.31) coincides with the Lagrangian, renormalized by the quantum fluctuations around the trivial vacuum (2.18), and corroborates the result obtained in [2].

We would like to emphasize that analysing the renormalization of the sine-Gordon model, caused by Gaussian fluctuations around a soliton, one can see that Gaussian fluctuations are perturbative fluctuations of order $O(\alpha_r(M^2)\beta^2)$ valid for $\beta^2 \ll 8\pi$, which cannot be responsible for non-perturbative contributions to the soliton mass at $\beta^2 = 8\pi$.

5. Renormalization of the soliton mass by Gaussian fluctuations in continuous space–time

Using the effective Lagrangian equation (4.24) one can calculate the soliton mass corrected by quantum fluctuations. It reads

$$M_s = \frac{8\sqrt{\alpha_0}}{\beta^2} + \Delta M_s, \tag{5.1}$$

where ΔM_s is defined by

$$\Delta M_s = -\int_{-\infty}^{+\infty}{\rm d}x^1\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x^1)] = \int_{-\infty}^{+\infty}\frac{{\rm d}k}{4\pi}\sqrt{\alpha_0 + k^2}\frac{{\rm d}\delta(k)}{{\rm d}k}. \tag{5.2}$$

This corresponds to the correction to the soliton mass, induced by Gaussian fluctuations, without a *surface term* $-\sqrt{\alpha_0}/\pi$ [19, 20].

In the Lorentz covariant form the correction to the soliton mass reads

$$\begin{aligned}\Delta M_s &= -\int_{-\infty}^{+\infty}{\rm d}x^1\,\delta\mathcal{L}_{\rm eff}[\vartheta_{c\ell}(x^1)]\\ &= \int_{-\infty}^{+\infty}\frac{{\rm d}\omega}{2\pi{\rm i}}\int_{-\infty}^{+\infty}\frac{{\rm d}k}{2\pi}\frac{\sqrt{\alpha_0}}{k^2+\alpha_0}[\ln(-\omega^2) - \ln(\alpha_0 - \omega^2 + k^2)]\\ &= -2\sqrt{\alpha_0}\int_{-\infty}^{+\infty}\frac{{\rm d}k}{2\pi}\int_{-\infty}^{+\infty}\frac{{\rm d}\omega}{2\pi{\rm i}}\frac{1}{\alpha_0 - \omega^2 + k^2 - {\rm i}0},\end{aligned} \tag{5.3}$$

where we have taken the effective Lagrangian defined by (4.22) and integrated over ω by parts. Using the result of the calculation of the integral (4.25) we get the following expression for the soliton mass corrected by Gaussian fluctuations

$$M_s = \frac{8\sqrt{\alpha_0(\Lambda^2)}}{\beta^2} - \frac{\sqrt{\alpha_0(\Lambda^2)}}{2\pi}\ln\left[\frac{\Lambda^2}{\alpha_0(\Lambda^2)}\right]. \tag{5.4}$$

The removal of the dependence on the cut-off Λ should be carried out within the renormalization procedure.

Replacing $\alpha_0(\Lambda^2)$ in (5.4) by $\alpha_r(M^2)Z_1(\beta^2, M^2; \Lambda^2)$, where the renormalization constant $Z_1(\beta^2, M^2; \Lambda^2)$ is defined by equation (1.4), we get

$$M_s = \frac{8\sqrt{\alpha_r(M^2)Z_1}}{\beta^2} - \frac{\sqrt{\alpha_r(M^2)Z_1}}{2\pi} \ln\left[\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\right]. \tag{5.5}$$

The renormalization constant Z_1 should be expanded in power of β^2 to order $O(\beta^2)$. This gives

$$Z_1 = 1 + \frac{\beta^2}{8\pi} \ln\left(\frac{\Lambda^2}{M^2}\right). \tag{5.6}$$

Substituting (5.6) into (5.5) and keeping only the leading terms in β^2 we get

$$M_s = \frac{8}{\beta^2}\sqrt{\alpha_r(M^2)\left[1 + \frac{\beta^2}{8\pi}\ln\left(\frac{\alpha_r(M^2)}{M^2}\right)\right]}. \tag{5.7}$$

Using equation (4.30) we can rewrite the rhs of (5.7) in terms of $\alpha_{\rm ph}$

$$M_s = \frac{8\sqrt{\alpha_{\rm ph}}}{\beta^2}. \tag{5.8}$$

The mass of a soliton M_s depends on the physical coupling constant $\alpha_{\rm ph}$. Hence, the contribution of Gaussian fluctuations around a soliton solution is absorbed by the renormalized coupling constant $\alpha_{\rm ph}$ and no singularities of the sine-Gordon model appear at $\beta^2 = 8\pi$.

This result confirms the assertion by Zamolodchikov and Zamolodchikov [14], that the singularity of the sine-Gordon model induced by the finite correction $-\sqrt{\alpha_{\rm ph}}/\pi$ to the soliton mass, caused by Gaussian fluctuations around a soliton solution, is completely due to the regularization and renormalization procedure. This has been corroborated in [2].

We have obtained that the soliton mass M_s does not depend on the normalization scale M. This testifies that the soliton mass M_s is an observable quantity.

6. Renormalization of soliton mass by Gaussian fluctuations: space–time discretization technique

Usually the correction to the soliton mass is investigated in the literature by a discretization procedure [19] (see also [20]). The soliton with Gaussian fluctuations is embedded into a spatial box with a finite volume L and various boundary conditions for Gaussian fluctuations at $x = \pm L/2$. In such a discretization approach time is also discrete with a period T, which finally has to be taken in the limit $T \to \infty$. The frequency spectrum is $\omega_m = 2\pi m/T$ with $m = 0, \pm 1, \ldots$. For various boundary conditions spectra of the momenta of Gaussian fluctuations around a soliton and of Klein–Gordon quanta, corresponding to vacuum fluctuations, are adduced in table 1. According to table 1, for various boundary conditions the corrections to the soliton mass are given by

$$\Delta M_s^{(\rm p)} = \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2{\rm i}T}\left\{2\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left[\ln\left(\alpha_0 - \omega_m^2 + k_n^2\right) - \ln\left(\alpha_0 - \omega_m^2 + q_n^2\right)\right]\right.$$
$$\left. + \sum_{m=-\infty}^{\infty}\left[\ln\left(-\omega_m^2 + \Delta^2(L)\right) - \ln\left(\alpha_0 - \omega_m^2\right)\right]\right\},$$

Table 1. The spectra of the momenta of Gaussian fluctuations around a soliton and the Klein–Gordon quanta. The modes, denoted by $(*)$ are due to the bound state.

Soliton sector		Vacuum sector
	Periodic BC	
$k_n L + \delta(k_n) = 2n\pi$		$q_n L = 2n\pi$
$n = 0 : (*)$	$\leftarrow 1\times \rightarrow$	$n = 0 : q_0 = 0$
$n = 1 : q_1 = \frac{\pi}{L} + \mathcal{O}(L^{-2})$	$\leftarrow 2\times \rightarrow$	$n = 1 : q_1 = \frac{2\pi}{L}$
$\cdots$	$\leftarrow 2\times \rightarrow$	$\cdots$
$\sum_{n=1}^{\infty} \rightarrow \int_{\frac{\pi}{L}+\mathcal{O}(L^{-2})}^{\infty} \frac{\mathrm{d}k}{2\pi}\left(L + \frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\right)$		$\sum_{n=1}^{\infty} \rightarrow \int_{\frac{2\pi}{L}}^{\infty} \frac{\mathrm{d}q}{2\pi} L$
	Anti-periodic BC	
$k_n L + \delta(k_n) = (2n-1)\pi$		$q_n L = (2n-1)\pi$
$n = 1 : (*) + k_1 = 0$	$\leftarrow (1+1)\times \rightarrow$	$n = 1 : q_1 = \frac{\pi}{L}$
$\cdots$	$\leftarrow 2\times \rightarrow$	$\cdots$
$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{2\pi}{L}+\mathcal{O}(L^{-2})}^{\infty} \frac{\mathrm{d}k}{2\pi}\left(L + \frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\right)$		$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{3\pi}{L}}^{\infty} \frac{\mathrm{d}q}{2\pi} L$
	Rigid walls	
$k_n L + \delta(k_n) = n\pi$		$q_n L = n\pi$
$n = 1 : (*)$	$\leftarrow 1\times \rightarrow$	$n = 1 : q_1 = \frac{\pi}{L}$
$\cdots$	$\leftarrow 1\times \rightarrow$	$\cdots$
$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{\pi}{L}+\mathcal{O}(L^{-2})}^{\infty} \frac{\mathrm{d}k}{\pi}\left(L + \frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\right)$		$\sum_{n=2}^{\infty} \rightarrow \int_{\frac{2\pi}{L}}^{\infty} \frac{\mathrm{d}q}{\pi} L$

$$\Delta M_s^{(\mathrm{ap})} = \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2\mathrm{i}T}\left\{2\sum_{m=-\infty}^{\infty}\sum_{n=2}^{\infty}\left[\ln\left(\alpha_0-\omega_m^2+k_n^2\right)-\ln\left(\alpha_0-\omega_m^2+q_n^2\right)\right]\right.$$
$$\left.+\sum_{m=-\infty}^{\infty}\left[\ln\left(-\omega_m^2+\Delta^2(L)\right)+\ln\left(\alpha_0-\omega_m^2\right)-2\ln\left(\alpha_0-\omega_m^2+q_1^2\right)\right]\right\},$$
$$\Delta M_s^{(\mathrm{rw})} = \lim_{T\to\infty}\lim_{L\to\infty}\frac{1}{2\mathrm{i}T}\left\{\sum_{m=-\infty}^{\infty}\sum_{n=2}^{\infty}\left[\ln\left(\alpha_0-\omega_m^2+k_n^2\right)-\ln\left(\alpha_0-\omega_m^2+q_n^2\right)\right]\right.$$
$$\left.+\sum_{m=-\infty}^{\infty}\left[\ln\left(-\omega_m^2+\Delta^2(L)\right)-\ln\left(\alpha_0-\omega_m^2+q_1^2\right)\right]\right\}, \qquad (6.1)$$

where $\Delta^2(L) \sim \alpha_0\,\mathrm{e}^{-\sqrt{\alpha_0}L}$ at $L \to \infty$, and the abbreviations (p), (ap) and (rw) mean periodic, anti-periodic boundary conditions and rigid walls, respectively.

For the summation over m, we use the formula, derived by Dolan and Jackiw [21]

$$\sum_{m=-\infty}^{+\infty}\left[\ln\left(-\omega_m^2+a^2\right)-\ln\left(-\omega_m^2+b^2\right)\right] = \mathrm{i}(a-b)T + 2\ln\left(\frac{1-\mathrm{e}^{-\mathrm{i}aT}}{1-\mathrm{e}^{-\mathrm{i}bT}}\right). \qquad (6.2)$$

Taking the limit $T \to \infty$ we arrive at the expression

$$\Delta M_s = -\frac{\sqrt{\alpha_0}}{2} + \lim_{L\to\infty}\begin{cases}\sum_{n=1}^{\infty}\left(\sqrt{\alpha_0+k_n^2}-\sqrt{\alpha_0+q_n^2}\right), & \text{periodic BC}\\ \sum_{n=2}^{\infty}\left(\sqrt{\alpha_0+k_n^2}-\sqrt{\alpha_0+q_n^2}\right), & \text{anti-periodic BC}\\ \frac{1}{2}\sum_{n=2}^{\infty}\left(\sqrt{\alpha_0+k_n^2}-\sqrt{\alpha_0+q_n^2}\right), & \text{rigid walls,}\end{cases} \qquad (6.3)$$

where BC is the abbreviation of *boundary conditions*.

The aim of our analysis of ΔM_s within the discretization procedure is to show that the discretization procedure gives ΔM_s in the form of (5.3).

The subsequent analysis of ΔM_s we carry out for periodic boundary conditions only. One can show that for anti-periodic boundary conditions and rigid walls the result is the same.

For the next transformation of the rhs of (6.3), we propose to use the following integral representation:

$$\sqrt{\alpha_0 + k_n^2} - \sqrt{\alpha_0 + q_n^2} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \ln\left[\frac{\alpha_0 + k_n^2 - \omega^2 - \mathrm{i}0}{\alpha_0 + q_n^2 - \omega^2 - \mathrm{i}0}\right]. \tag{6.4}$$

This gives after interchanging the integration over ω with the summation over n

$$\Delta M_s^{(\mathrm{p})} = -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty}\lim_{N\to\infty} \sum_{n=1}^{N} \ln\left[\frac{\alpha_0 + k_n^2 - \omega^2 - \mathrm{i}0}{\alpha_0 + q_n^2 - \omega^2 - \mathrm{i}0}\right], \tag{6.5}$$

with *mode-counting* regularization procedure [20] applied. Passing to the continuous momentum representation [20] and using the spectra of the momenta of Gaussian fluctuations and the Klein–Gordon fluctuations (the vacuum fluctuations) adduced in table 1 we transcribe the rhs of (6.5) into the form

$$\begin{aligned}
\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty}\lim_{N\to\infty} \left[\int_{k_1}^{k_N} \mathrm{d}k \frac{\mathrm{d}n(k)}{\mathrm{d}k} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0)\right. \\
&\quad \left. - \int_{q_1}^{q_N} \mathrm{d}q \frac{\mathrm{d}n(q)}{\mathrm{d}q} \ln(\alpha_0 + q^2 - \omega^2 - \mathrm{i}0)\right] \\
&= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty}\lim_{N\to\infty} \left[\int_{k_1}^{k_N} \frac{\mathrm{d}k}{2\pi}\left(L + \frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\right)\right. \\
&\quad \left. \times \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) - L\int_{q_1}^{q_N} \frac{\mathrm{d}k}{2\pi} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0)\right],
\end{aligned} \tag{6.6}$$

where the limits are equal to (see table 1)

$$\begin{aligned}
k_1 &= \frac{\pi}{L}, \qquad & k_N &= q_N - \frac{\delta(q_N)}{L} = q_N - \frac{\sqrt{\alpha_0}}{\pi N}, \\
q_1 &= \frac{2\pi}{L}, \qquad & q_N &= \frac{2\pi N}{L}.
\end{aligned} \tag{6.7}$$

Rearranging the limits of integrations we get

$$\begin{aligned}
\Delta M_s^{(\mathrm{p})} &= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty}\lim_{N\to\infty} \int_{\pi/L}^{k_N} \frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) \\
&\quad + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty} L\int_{\pi/L}^{2\pi/L} \frac{\mathrm{d}k}{2\pi} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) \\
&\quad - \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty}\lim_{N\to\infty} L\int_{k_N}^{q_N} \frac{\mathrm{d}k}{2\pi} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) \\
&= -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty} \int_{\pi/L}^{\infty} \frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) \\
&\quad + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty} L\int_{\pi/L}^{2\pi/L} \frac{\mathrm{d}k}{2\pi} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) \\
&\quad - \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty}\lim_{N\to\infty} L\frac{k_N - q_N}{2\pi} \ln(\alpha_0 + k^{*2} - \omega^2 - \mathrm{i}0).
\end{aligned} \tag{6.8}$$

For the last term we have applied the *mean value theorem* with $q_N - \delta(q_N)/L < k^* < q_N$. Since the difference $k_N - q_N = \delta(q_N)/L = \sqrt{\alpha_0}/\pi N$ is of order $O(1/N)$, this term vanishes in the limit $N \to \infty$.[7] As a result we get

$$\Delta M_s^{(\mathrm{p})} = -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty} \int_{\pi/L}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{d}\delta(k)}{\mathrm{d}k} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \lim_{L\to\infty} L \int_{\pi/L}^{2\pi/L} \frac{\mathrm{d}k}{2\pi} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0). \quad (6.9)$$

Taking the limit $L \to \infty$ and applying to the last term the *mean value theorem* we arrive at the expression

$$\Delta M_s^{(\mathrm{p})} = -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{4\pi\mathrm{i}} \ln(\alpha_0 - \omega^2 - \mathrm{i}0) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_0^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{d}\delta(k)}{\mathrm{d}k} \ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0), \quad (6.10)$$

which we transcribe into the form

$$\Delta M_s^{(\mathrm{p})} = -\frac{\sqrt{\alpha_0}}{2} + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_0^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{i}\delta(k)}{\mathrm{d}k} \ln(-\omega^2 - \mathrm{i}0) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{4\pi\mathrm{i}} \ln(\alpha_0 - \omega^2 - \mathrm{i}0) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_0^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{d}\delta(k)}{\mathrm{d}k} [\ln(\alpha_0 + k^2 - \omega^2 - \mathrm{i}0) - \ln(-\omega^2 - \mathrm{i}0)]. \quad (6.11)$$

The next steps of the reduction of the rhs of (6.11) to the form (5.3) are rather straightforward. First, one can easily shows that

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \int_0^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{d}\delta(k)}{\mathrm{d}k} \ln(-\omega^2 - \mathrm{i}0) + \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{4\pi\mathrm{i}} \ln(\alpha_0 - \omega^2 - \mathrm{i}0) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{4\pi\mathrm{i}} [\ln(\alpha_0 - \omega^2 - \mathrm{i}0) - \ln(-\omega^2 - \mathrm{i}0)] = \frac{\sqrt{\alpha_0}}{2} \quad (6.12)$$

and, second, integrating over ω by parts the last integral in (6.11) can be reduced to the form

$$\Delta M_s^{(\mathrm{p})} = -2\sqrt{\alpha_0} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{1}{\alpha_0 - \omega^2 + k^2 - \mathrm{i}0}. \quad (6.13)$$

Thus, we have shown that the correction to the soliton mass, induced by Gaussian fluctuations around a soliton and calculated by means of the discretization procedure, agrees fully with that we have obtained in continuous space–time (5.3).

Hence, the renormalization of the soliton mass, caused by Gaussian fluctuations calculated within the space–time discretization technique, coincides with the renormalization of the soliton mass in continuous space–time. We would like to emphasize that the obtained result (6.13) does not depend on the boundary conditions.

The calculation of the functional determinant within the discretization procedure has confirmed the absence of the correction $-\sqrt{\alpha_0}/\pi$. This agrees with our assertion that such a correction does not appear due to Gaussian fluctuations around a soliton, corresponding to quantum fluctuations to first orders in $\alpha_0(\Lambda^2)$ and β^2.

The reduction of $\Delta M_s^{(\mathrm{p})}$ of equation (6.1) to expression (5.3) can be carried out directly. First, summing over n within the *mode-counting* regularization procedure and taking the limit

[7] We would like to emphasize that exactly the term of this kind leads to the contribution of the finite *surface term* $-\sqrt{\alpha_0}/\pi$ in a regularization procedure using expressions (6.3) without turning to the integral representation (6.4).

$L \to \infty$ we arrive from $\Delta M_s^{(p)}$ of equation (6.1) at

$$\begin{aligned}
\Delta M_s^{(p)} &= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\lim_{L\to\infty}\left\{\frac{1}{2}\left[\ln\left(-\omega_m^2+\Delta^2(L)\right)-\ln\left(\alpha_0-\omega_m^2\right)\right]\right.\\
&\quad\left.+\lim_{N\to\infty}\sum_{n=1}^{N}\left[\ln\left(\alpha_0+k_n^2-\omega_m^2\right)-\ln\left(\alpha_0+q_n^2-\omega_m^2\right)\right]\right\}\\
&= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\lim_{L\to\infty}\left\{\frac{1}{2}\left[\ln\left(-\omega_m^2+\Delta^2(L)\right)-\ln\left(\alpha_0-\omega_m^2\right)\right]\right.\\
&\quad\left.+\lim_{N\to\infty}\left[\int_{k_1}^{k_N}\mathrm{d}k\frac{\mathrm{d}n(k)}{\mathrm{d}k}\ln\left(\alpha_0+k^2-\omega_m^2\right)-\int_{q_1}^{q_N}\mathrm{d}q\frac{\mathrm{d}n(q)}{\mathrm{d}q}\ln\left(\alpha_0+q^2-\omega_m^2\right)\right]\right\}\\
&= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\lim_{L\to\infty}\left\{\frac{1}{2}\left[\ln\left(-\omega_m^2+\Delta^2(L)\right)-\ln\left(\alpha_0-\omega_m^2\right)\right]\right.\\
&\quad+\lim_{N\to\infty}\left[\int_{\pi/L}^{k_N}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\ln\left(\alpha_0+k^2-\omega_m^2\right)+L\int_{\pi/L}^{2\pi/L}\frac{\mathrm{d}k}{2\pi}\ln\left(\alpha_0+k^2-\omega_m^2\right)\right.\\
&\quad\left.\left.-L\int_{k_N}^{q_N}\frac{\mathrm{d}k}{2\pi}\ln\left(\alpha_0-\omega_m^2+k^2\right)\right]\right\}\\
&= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\lim_{L\to\infty}\left\{\frac{1}{2}\left[\ln\left(-\omega_m^2+\Delta^2(L)\right)-\ln\left(\alpha_0-\omega_m^2\right)\right]\right.\\
&\quad\left.+\int_{\pi/L}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\ln\left(\alpha_0+k^2-\omega_m^2\right)+L\int_{\pi/L}^{2\pi/L}\frac{\mathrm{d}k}{2\pi}\ln\left(\alpha_0+k^2-\omega_m^2\right)\right\}\\
&= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\left\{\frac{1}{2}\left[\ln\left(-\omega_m^2\right)-\ln\left(\alpha_0-\omega_m^2\right)\right]\right.\\
&\quad\left.+\int_{0}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\ln\left(\alpha_0+k^2-\omega_m^2\right)+\frac{1}{2}\ln\left(\alpha_0-\omega_m^2\right)\right\}\\
&= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\left\{\int_{0}^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\delta(k)}{\mathrm{d}k}\ln\left(\alpha_0+k^2-\omega_m^2\right)+\frac{1}{2}\ln\left(-\omega_m^2\right)\right\}.
\end{aligned}\tag{6.14}$$

Now we use the integral representation [5] and get

$$\begin{aligned}
\Delta M_S^{(p)} &= \lim_{T\to\infty}\frac{1}{\mathrm{i}T}\sum_{m=-\infty}^{\infty}\int_0^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{2\sqrt{\alpha_0}}{\alpha_0+k^2}\left[\ln\left(-\omega_m^2\right)-\ln\left(\alpha_0+k^2-\omega_m^2\right)\right]\\
&= \lim_{T\to\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{\mathrm{i}T}\frac{\mathrm{d}m(\omega)}{\mathrm{d}\omega}\int_0^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{2\sqrt{\alpha_0}}{\alpha_0+k^2}[\ln(-\omega^2)-\ln(\alpha_0+k^2-\omega^2)]\\
&= \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_0^{\infty}\frac{\mathrm{d}k}{2\pi}\frac{2\sqrt{\alpha_0}}{\alpha_0+k^2}[\ln(-\omega^2)-\ln(\alpha_0+k^2-\omega^2)]\\
&= -2\sqrt{\alpha_0}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\frac{1}{\alpha_0-\omega^2+k^2-\mathrm{i}0}.
\end{aligned}\tag{6.15}$$

For the other boundary conditions we get the same result.

Thus, we have shown that the discretized version of the correction to the soliton mass reduces to the continuum result if one transcribes first the sum over the quantum number n of the momenta of Gaussian and vacuum fluctuations into the corresponding integral over the momentum k.

7. Renormalization of the two-point Green function in the massive sine-Gordon model

In this section, we show that the renormalization procedure of the two-point Green function in the sine-Gordon model, developed above, can be applied to the renormalization of the two-point Green function in the massive sine-Gordon model (the MSG model) [12, 23–34]. The *bare* Lagrangian of the MSG model is [12]

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}m_0^2(\Lambda^2)\vartheta^2(x) + \frac{\alpha_0(\Lambda^2)}{\beta^2}(\cos\beta\vartheta(x) - 1), \tag{7.1}$$

where $m_0(\Lambda^2)$ is the *bare* mass of the free sine-Gordon quanta. The renormalized Lagrangian reads

$$\begin{aligned}\mathcal{L}(x) &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}m_r^2(M^2)\vartheta^2(x) + \frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1)\\ &\quad - \frac{1}{2}m_r^2(M^2)(Z_m - 1)\vartheta^2(x) + (Z_1 - 1)\frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1)\\ &= \frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}Z_m m_r^2(M^2)\vartheta^2(x) + Z_1\frac{\alpha_r(M^2)}{\beta^2}(\cos\beta\vartheta(x) - 1),\end{aligned} \tag{7.2}$$

where $Z_1 = Z_1(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)$ is defined by (1.4). It relates the renormalized coupling constant $\alpha_r(M^2)$ to the *bare* coupling constant $\alpha_0(\Lambda^2)$ through the relation (1.3). Then, $Z_m = Z_m(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)$ is the renormalization constant of the mass of the MSG model field

$$m_r(M^2) = Z_m^{-1/2}(\alpha_r(M^2), \beta^2, M^2; \Lambda^2)m_0(\Lambda^2). \tag{7.3}$$

Similar to the sine-Gordon model, we keep the coupling constant β^2 unrenormalizable.

For the analysis of the renormalizability of the MSG model with respect to quantum fluctuations around the trivial vacuum, we expand the Lagrangian (7.2) in powers of $\vartheta(x)$. This gives

$$\mathcal{L}(x) = \tfrac{1}{2}\left[\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \tilde{m}_r^2(M^2)\vartheta^2(x)\right] + \mathcal{L}_{\rm int}(x), \tag{7.4}$$

where $\tilde{m}_r^2(M^2) = m_r^2(M^2) + \alpha_r(M^2)$ is the effective mass of the sine-Gordon quanta, $\mathcal{L}_{\rm int}(x)$ describes self-interactions of the MSG model field

$$\begin{aligned}\mathcal{L}_{\rm int}(x) &= -\frac{1}{2}m_r^2(M^2)(Z_m - 1)\vartheta^2(x) + \alpha_r(M^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x)\\ &\quad + (Z_1 - 1)\alpha_r(M^2)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x).\end{aligned} \tag{7.5}$$

It is seen that the parameter $\tilde{m}_r(M^2)$ has the meaning of a mass of free quanta of the MSG model field $\vartheta(x)$. The causal two-point Green function of free MSG model quanta with mass $\tilde{m}_r(M^2)$ is defined by

$$-{\rm i}\Delta_F\big(x; \tilde{m}_r^2(M^2)\big) = \langle 0|{\rm T}(\vartheta(x)\vartheta(0))|0\rangle = \int\frac{{\rm d}^2k}{(2\pi)^2{\rm i}}\frac{{\rm e}^{-{\rm i}k\cdot x}}{\tilde{m}_r^2(M^2) - k^2 - {\rm i}0}. \tag{7.6}$$

At $x = 0$ the Green function $-{\rm i}\Delta_F(0; \tilde{m}_r^2(M^2))$ is equal to [2]

$$-{\rm i}\Delta_F\big(0; \tilde{m}_r^2(M^2)\big) = \frac{1}{4\pi}\ln\left[\frac{\Lambda^2}{\tilde{m}_r^2(M^2)}\right], \tag{7.7}$$

where Λ is a cut-off in the Euclidean two-dimensional momentum space.

The calculation of the correction to the two-point Green function in the MSG model to first order in $\alpha_r(M^2)$ and to all orders in β^2 runs parallel to that by (2.10). The two-point Green function reads

$$
\begin{aligned}
-\mathrm{i}\Delta(x) &= -\mathrm{i}\Delta\big(x, \tilde{m}_r^2(M^2)\big) + \mathrm{i}\alpha_r(M^2)\left[1 - Z_1 \exp\left\{\frac{1}{2}\beta^2 \mathrm{i}\Delta_F\big(0, \tilde{m}_r^2(M^2)\big)\right\}\right] \\
&\quad \times \int \mathrm{d}^2 y\big[-\mathrm{i}\Delta_F\big(x-y, \tilde{m}_r^2(M^2)\big)\big]\big[-\mathrm{i}\Delta_F\big(-y, \tilde{m}_r^2(M^2)\big)\big] \\
&\quad + (-\mathrm{i})(Z_m - 1)m_r^2(M^2) \int \mathrm{d}^2 y\big[-\mathrm{i}\Delta_F\big(x-y, \tilde{m}_r^2(M^2)\big)\big]\big[-\mathrm{i}\Delta_F\big(-y, \tilde{m}_r^2(M^2)\big)\big].
\end{aligned} \tag{7.8}
$$

In the momentum representation, this expression takes the form

$$
\begin{aligned}
-\mathrm{i}\tilde{\Delta}(p) &= \frac{(-\mathrm{i})}{\tilde{m}_r^2(M^2) - p^2} + \frac{(-\mathrm{i})}{\tilde{m}_r^2(M^2) - p^2}\mathrm{i}\alpha_r(M^2)\left[1 - \left(\frac{\tilde{m}_r^2(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]\frac{(-\mathrm{i})}{\tilde{m}_r^2(M^2) - p^2} \\
&\quad + \frac{(-\mathrm{i})}{\tilde{m}_r^2(M^2) - p^2}(-\mathrm{i})(Z_m - 1)m_r^2(M^2)\frac{(-\mathrm{i})}{\tilde{m}_r^2(M^2) - p^2}.
\end{aligned} \tag{7.9}
$$

The two last terms define the correction to the mass of the MSG model field

$$
\delta m_r^2(M^2) = -\alpha_r(M^2)\left[1 - \left(\frac{\tilde{m}_r^2(M^2)}{M^2}\right)^{\beta^2/8\pi}\right] + (Z_m - 1)m_r^2(M^2). \tag{7.10}
$$

Thus, for the two-point Green function, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2, we get

$$
-\mathrm{i}\tilde{\Delta}(p) = \frac{(-\mathrm{i})}{\tilde{m}_r^2(M^2) + \delta m_r^2(M^2) - p^2} = \frac{(-\mathrm{i})}{m_{\mathrm{ph}}^2 - p^2}, \tag{7.11}
$$

where m_{ph}, the physical mass of the MSG model quanta, is determined by

$$
\begin{aligned}
m_{\mathrm{ph}}^2 &= \tilde{m}_r^2(M^2) - \alpha_r(M^2)\left[1 - \left(\frac{m_r^2(M^2) + \alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right] + (Z_m - 1)m_r^2(M^2) \\
&= m_r^2(M^2) + \alpha_r(M^2)\left(\frac{m_r^2(M^2) + \alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi} + (Z_m - 1)m_r^2(M^2).
\end{aligned} \tag{7.12}
$$

Since the first two terms do not depend on the cut-off Λ, the last term in (7.12) should vanish. This means that the renormalization constant $(Z_m - 1)$ is of order $O\big(\alpha_r^2(M^2)\big)$, i.e. $(Z_m - 1)m_r^2(M^2) = 0$ to first order in $\alpha_r(M^2)$ and to all orders in β^2. Thus, the squared physical mass of the MSG model field is

$$
m_{\mathrm{ph}}^2 = m_r^2(M^2) + \alpha_r(M^2)\left(\frac{m_r^2(M^2) + \alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}. \tag{7.13}
$$

In the soft-boson limit, when $m_r^2(M^2) \to 0$, the physical mass of the MSG model field coincides with (2.16).

This agrees with the assertion that the sine-Gordon model is not infrared singular [2] and testifies that the operator $m_0^2\vartheta^2(x)$ is *soft*. That is in agreement with the results obtained by Amit *et al* [12].

For finite $m_r^2(M^2)$ and in the perturbative regime $m_r^2(M^2) \gg \alpha_r(M^2)$ the physical mass of the MSG model field is equal to

$$m_{\rm ph}^2 = m_r^2(M^2) + \alpha_r(M^2)\left(\frac{m_r^2(M^2)}{M^2}\right)^{\beta^2/8\pi}, \tag{7.14}$$

where we have kept only the leading terms in $\alpha_r(M^2)$ expansion.

Calculating the second-order correction to the two-point Green function one can show that the renormalization constant $(Z_m - 1)$ vanishes. This implies that the mass parameter $m_0(\Lambda^2)$ is unrenormalizable, i.e. $m_0(\Lambda^2) = m_0$. In this case, the physical mass of the MSG model field takes the form

$$m_{\rm ph}^2 = m_0^2 + \alpha_r(M^2)\left(\frac{m_0^2}{M^2}\right)^{\beta^2/8\pi}. \tag{7.15}$$

Since the physical mass of the MSG model field cannot depend on the normalization scale, we have to set

$$\alpha_{\rm ph} = \alpha_r(M^2)\left(\frac{m_0^2}{M^2}\right)^{\beta^2/8\pi} \quad\longrightarrow\quad \alpha_r(M^2) = \alpha_{\rm ph}\left(\frac{M^2}{m_0^2}\right)^{\beta^2/8\pi}. \tag{7.16}$$

It is seen that setting the normalization scale $M = m_0$ the renormalized coupling constant $\alpha_r\big(m_0^2\big)$ coincides with the physical one, i.e. $\alpha_r\big(m_0^2\big) = \alpha_{\rm ph}$.

The Gell-Mann–Low function, calculated for the coupling constant $\alpha_r(M^2)$ defined in (7.16), is equal to

$$M\frac{\partial\alpha_r(M^2)}{\partial M} = \beta(\alpha_r(M^2), \beta^2) = \frac{\beta^2}{4\pi}\alpha_r(M^2). \tag{7.17}$$

The Callan–Symanzik equation for the two-point Green function of the MSG model field reads

$$\left[p^2\frac{\partial}{\partial p^2} - \delta(\beta^2)\alpha_r(M^2)\frac{\partial}{\partial\alpha_r(M^2)} - \frac{1}{2}m_0^2\frac{\partial}{\partial m_0^2} + 1\right]\tilde\Delta\big(p^2;\alpha_r(M^2),\beta^2,m_0^2\big) = 0, \tag{7.18}$$

where following [12] we have denoted $\delta(\beta^2) = (\beta^2 - 8\pi)/8\pi$. For the derivation of this equation we have used

$$F(0,p;\alpha_r(M^2),\beta^2) = \left(-m_0^2\frac{\partial}{\partial m_0^2} + 2\alpha_r(M^2)\frac{\partial}{\partial\alpha_r(M^2)}\right)\tilde\Delta\big(p;\alpha_r(M^2),\beta^2,m_0^2\big). \tag{7.19}$$

Setting $\tilde\Delta\big(p^2;\alpha_r(M^2),\beta^2,m_0^2\big) = D\big(p^2;\alpha_r(M^2),\beta^2,m_0^2\big)\big/p^2$ and introducing dimensionless variables $t = p^2m_0^2\big/M^4$, $\tilde\alpha_r = \alpha_r(M^2)/M^2$ we get

$$\left[t\frac{\partial}{\partial t} - 2\delta(\beta^2)\tilde\alpha\frac{\partial}{\partial\tilde\alpha}\right]D(t;\tilde\alpha) = 0. \tag{7.20}$$

This agrees well with the renormalization group equation obtained by Amit *et al* [12], where $2\delta(\beta^2)\tilde\alpha$ is the Gell-Mann–Low function, calculated to first order in $\tilde\alpha$ and to all orders in β^2.

According to the general theory of partial differential equations of first order [15], the solution of (7.20) is an arbitrary function of the integration constant

$$C = \tilde\alpha t^{2\delta(\beta^2)}, \tag{7.21}$$

which is the solution of the characteristic differential equation

$$\frac{{\rm d}t}{t} = \frac{{\rm d}\tilde\alpha}{-2\delta(\beta^2)\tilde\alpha}. \tag{7.22}$$

Hence, the Fourier transform of the two-point Green function of the sine-Gordon field is equal to

$$\tilde\Delta\big(p^2;\alpha_{\rm ph},\beta^2,m_0^2\big) = \frac{1}{p^2}D\left[\frac{\alpha_{\rm ph}}{m_0^2}\left(\frac{M^2}{m_0^2}\right)^{\delta(\beta^2)}\left(\frac{p^2m_0^2}{M^4}\right)^{2\delta(\beta^2)}\right]. \tag{7.23}$$

Making the renormalization at $M^2 = m_0^2$ we get

$$\tilde{\Delta}\big(p^2; \alpha_{\rm ph}, \beta^2, m_0^2\big) = \frac{1}{p^2} D\left[\frac{\alpha_r(p^2)}{m_0^2}\right], \tag{7.24}$$

where, in analogy with our analysis in section 3, we have introduced the running coupling constant $\alpha_r(p^2)$

$$\alpha_r(p^2) = \alpha_{\rm ph}\left(\frac{p^2}{m_0^2}\right)^{2\delta(\beta^2)}. \tag{7.25}$$

For $\delta(\beta^2) < 0$, i.e. $\beta^2 < 8\pi$, the MSG model with quantum fluctuations around a trivial vacuum, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2, is an asymptotically free theory for $p^2 \to \infty$. In turn for $\delta(\beta^2) > 0$, i.e. $\beta^2 > 8\pi$, the running coupling constant $\alpha_r(p^2)$ grows with p^2. Of course, due to a perturbative derivation of the Gell-Mann–Low function (7.17) and the Callan–Symanzik equation (7.18), the running coupling constant $\alpha_r(p^2)$ cannot grow to infinity. The allowed region for momenta p^2 is restricted by the inequality $\alpha_r(p^2) \ll m_0^2$. This gives

$$p^2 \ll m_0^2\left(\frac{m_0^2}{\alpha_{\rm ph}}\right)^{1/2\delta(\beta^2)}. \tag{7.26}$$

Thus, we have shown that our results on the renormalization of the massive sine-Gordon model, carried out for the two-point Green function, agree well with those obtained by Amit *et al* [12].

8. Conclusion

We have investigated the renormalizability of the sine-Gordon model. We have analysed the renormalizability of the two-point Green function to second order in α and to all orders in β^2. We have shown that the divergences appearing in the sine-Gordon model can be removed by the renormalization of the dimensional coupling constant $\alpha_0(\Lambda^2)$. We recall that the coupling constant β^2 is not renormalizable. This agrees well with a possible interpretation of the coupling constant β^2 as $\hbar$ [1, 22]. The perturbation theory is developed with respect to the renormalized dimensional coupling constant $\alpha_r(M^2)$ depending on the normalization scale M and the dimensionless coupling constant β^2. Quantum fluctuations relative to the trivial vacuum calculated to first order in $\alpha_r(M^2)$ and to arbitrary order in β^2 form a physical coupling constant $\alpha_{\rm ph}$ after the removal of divergences. The physical coupling constant $\alpha_{\rm ph}$ is finite and does not depend on the normalization scale M. We have argued that the total renormalized two-point Green function depends on the physical coupling constant $\alpha_{\rm ph}$ only. In order to illustrate this assertion (i) we have calculated the correction to the two-point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2 and (ii) we have solved the Callan–Symanzik equation for the two-point Green function with the Gell-Mann–Low function, defined to all orders in $\alpha_r(M^2)$ and β^2. We have found that the two-point Green function of the sine-Gordon field depends on the running coupling constant $\alpha_r(p^2) = \alpha_{\rm ph}(p^2/\alpha_{\rm ph})^{\tilde{\beta}^2/8\pi}$, where $\tilde{\beta}^2 = \beta^2/(1+\beta^2/8\pi) < 1$ for any β^2.

In addition to the analysis of the renormalizability of the sine-Gordon model with respect to quantum fluctuations relative to the trivial vacuum, we have analysed the renormalizability of the sine-Gordon model with respect to quantum fluctuations around a soliton. Following Dashen *et al* [9, 10] and Faddeev and Korepin [11] we have taken into account only Gaussian fluctuations.

For the calculation of the effective Lagrangian, induced by Gaussian fluctuations, we have used the path-integral approach and integrated over the field $\varphi(x)$, fluctuating around a soliton. This has allowed us to express the effective Lagrangian in terms of the functional determinant. For the calculation of the contribution of the functional determinant we have used the eigenfunctions and eigenvalues of the differential operator, describing the evolution of the field $\varphi(x)$. We have shown that the renormalized effective Lagrangian, induced by Gaussian fluctuations around a soliton, coincides completely with the renormalized Lagrangian of the sine-Gordon model, caused by quantum fluctuations around the trivial vacuum to first order in α_0 and to second order in β^2. After the removal of divergences the soliton mass is equal to the mass of a soliton, calculated without quantum corrections, up to the replacement $\alpha_0 \to \alpha_{\rm ph}$. This implies that Gaussian fluctuations around a soliton do not produce any quantum corrections to the soliton mass. Hence, no non-perturbative singularities of the sine-Gordon model at $\beta^2 = 8\pi$ can be induced by Gaussian fluctuations around a soliton.

For the confirmation of our results, obtained in continuous space–time, we have calculated the functional determinant caused by Gaussian fluctuations around a soliton within the discretization procedure with periodic and anti-periodic boundary conditions and rigid walls. We have shown that the result of the calculation of the functional determinant (i) coincides with that obtained in continuous space–time and (ii) does not depend on the boundary conditions.

Finally, we have analysed the renormalization of the two-point Green function of the massive sine-Gordon model. We have shown that the mass operator $m_0^2\vartheta^2(x)$ is soft. In the infrared limit $m_0 \to 0$ the physical mass of the massive sine-Gordon model quanta reduces to our result (2.16). For $m_0^2 \gg \alpha_r(M^2)$ we have shown that the mass parameter m_0 is unrenormalizable. The physical coupling constant $\alpha_{\rm ph}$ has been calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2. This has allowed us to calculate the Gell-Mann–Low function and to derive the Callan–Symanzik equation for the two-point Green function. We have shown that the Callan–Symanzik equation reduces to the form used by Amit *et al* [12] to the same order in perturbation theory. Solving this equation we have calculated the running coupling constant and found that for $\beta^2 < 8\pi$ the massive sine-Gordon model with quantum fluctuations around a trivial vacuum, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2, is asymptotically free for infinitely large momenta. In turn, for $\beta^2 > 8\pi$ the running coupling constant $\alpha_r(p^2)$ grows with p^2. But since $\alpha_r(p^2)$ has been calculated perturbatively for $m_0^2 \gg \alpha_r(M^2)$, the running coupling constant should obey the constraint $m_0^2 \gg \alpha_r(p^2)$. This restricts the region of the allowed momenta $p^2 \ll m_0^2\big(m_0^2/\alpha_{\rm ph}\big)^{1/2\delta(\beta^2)}$ with $\delta(\beta^2) = (\beta^2 - 8\pi)/8\pi$ [12]. All these results do not contradict those obtained by Amit *et al* [12].

The renormalization of the sine-Gordon model, which was carried out before 1979 in [23–26], has been discussed perfectly well by Amit *et al*. After 1980, as has been pointed out by Nándori *et al* [27], the main results on the renormalization of the sine-Gordon model in two dimensions have been obtained in [28–34]. In these papers, the sine-Gordon model has been investigated at finite temperature in connection with the XY model and the existence of phase transitions. The Coleman fixed point at $\beta^2 = 8\pi$ has been recovered in our approach [2] as well as in the present paper.

Unlike [12, 23–34], we would like to apply the results obtained in this paper to the analysis of the FQHE [6, 7]. As has been shown in [22], the massive Thirring model, which can describe one-dimensional edge fermions [6, 7], bosonises to the sine-Gordon model for $\beta^2 > 8\pi$. According to [22], for $\beta^2 > 8\pi$ the sine-Gordon system produces mainly solitons, which can play an important role in the FQHE [35].

References

[1] Coleman S 1975 *Phys. Rev.* D **11** 2088
[2] Faber M and Ivanov A N 2003 *J. Phys. A: Math. Gen.* **36** 7839, and references therein

[3] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[4] Peskin M E and Schroeder D V 1995 *Quantum Field Theory* (Reading, MA: Perseus Books)
[5] Weinberg S 1996 The quantum theory of fields *Modern Applications* vol II (Cambridge: Cambridge University Press)
[6] Ezawa Z F 2000 Quantum Hall effects *Field Theoretical Approach and Related Topics* (Singapore: World Scientific) pp 299–304
[7] Ilieva N and Thirring W 2001 *Eur. Phys. J.* C **19** 561
[8] Ezawa Z F 2000 Quantum Hall effects *Field Theoretical Approach and Related Topics* (Singapore: World Scientific) p 139
[9] Dashen R F, Hasslacher B and Neveu A 1974 *Phys. Rev.* D **10** 4130
[10] Dashen R F, Hasslacher B and Neveu A 1975 *Phys. Rev.* D **11** 3424
[11] Faddeev L D and Korepin V E 1978 *Phys. Rep.* **42** 1
[12] Amit D, Goldschmidt Y Y and Grinstein G 1980 *J. Phys. A: Math. Gen.* **13** 585
[13] Bogoliubov N N and Shirkov D V 1959 *Introduction to the Quantum Theory of Quantized Fields* (New York: Interscience)
[14] Zamolodchikov Alexander B and Zamolodchikov Alexey B 1979 *Ann. Phys.* **120** 253
[15] Courant R and Hilbert D 1962 Methods of mathematical physics *Partial Differential Equations* vol II (New York: Interscience/Wiley)
[16] Rubinstein J 1970 *J. Math. Phys.* **11** 258
[17] Lebedev N N, Slalskaya I P and Uflyand Y S 1965 *Problems of Mathematical Physics* (Englewood Cliffs, NJ: Prentice-Hall) pp 55–102
[18] Peskin M E and Schroeder D V 1995 *Quantum Field Theory* (Reading, MA: Perseus Books) p 286
[19] Rajaraman R 1982 *Solitons and Instantons* (Amesterdam: North-Holland)
[20] Rebhan A and van Nieuwenhuizen P 1997 *Nucl. Phys.* B **508** 449
[21] Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 3320
[22] Faber M and Ivanov A N 2001 *Eur. Phys. J.* C **20** 723 (*Preprint* hep-th/0105057)
[23] Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 118
Kosterlitz J M 1974 *J. Phys. C: Solid State Phys.* **7** 1046
[24] Jose J V, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
[25] Wiegmann P B 1978 *J. Phys. C: Solid State Phys.* **11** 1583
[26] Samuel S 1978 *Phys. Rev.* D **18** 1916
[27] Nándori I, Sailer K, Jentschura U D and Soff G 2004 *Phys. Rev.* D **69** 025004
[28] Nienhuis B 1987 *Phase Transitions and Critical Phenomena* vol 11 ed C Domb and J L Lebowitz (London: Academic) pp 1–53
[29] Huang K and Polonyi J 1991 *Int. Mod. Phys.* A **6** 409
[30] Creswick R J, Farach H A and Poole C P Jr 1998 *Introduction to RG Methods in Physics* (New York: Wiley)
[31] Gulácsi Zs and Gulácsi M 1998 *Adv. Phys.* **47** 1
[32] von Gersdorf G and Wetterich C 2001 *Phys. Rev.* B **64** 054513
[33] Nándori I, Polonyi J and Sailer K 2001 *Phys. Rev.* D **63** 045022
Nándori I, Polonyi J and Sailer K 2001 *Phil. Mag.* B **81** 1615
Nándori I, Sailer K, Jentschura U D and Soff G 2002 *J. Phys. G: Nucl. Part. Phys.* **28** 607
[34] Fertig H A and Majumdar K 2003 *Preprint* cond-mat/0302012
[35] Bernevig B A, Brodie J H, Susskind L and Toumbas N 2001 *J. High Energy Phys.* JHEP02(2001)003 (*Preprint* hep-th/0010105)
Bena I and Nudelman A 2000 *J. High Energy Phys.* JHEP12(2000)017